

The Large N 't Hooft Limit of Coset Minimal Models

Changhyun Ahn ¹

Department of Physics, Princeton University, Jadwin Hall, Princeton, NJ 08544, USA
Department of Physics, Kyungpook National University, Taegu 702-701, Korea

ahn@knu.ac.kr

Abstract

Recently, Gaberdiel and Gopakumar proposed that the two-dimensional WA_{N-1} minimal model conformal field theory in the large N 't Hooft limit is dual to the higher spin theories on the three-dimensional AdS space with two complex scalars. In this paper, we examine this proposal for the $WD_{\frac{N}{2}}$ and $WB_{\frac{N-1}{2}}$ minimal models initiated by Fateev and Lukyanov in 1988. By analyzing the renormalization group flows on these models, we find that the gravity duals in AdS space are higher spin theories coupled to two equally massive real scalar fields. We also describe the large N 't Hooft limit for the minimal model of the second parafermion theory.

¹On leave from the Department of Physics, Kyungpook National University, Taegu 702-701, Korea and address until Aug. 31, 2011: Department of Physics, Princeton University, Jadwin Hall, Princeton, NJ 08544, USA

1 Introduction

By looking at the well understood family of two-dimensional conformal field theories with an appropriate large N limit, Gaberdiel and Gopakumar [1, 2] have been using the AdS/CFT correspondence to look for three-dimensional classical gravity theories. They consider a particular A_{N-1} WZW coset minimal model [3, 4, 5, 6] which has a higher spin $WA_{N-1}(\equiv W_N)$ symmetry generated by currents of spins $2, 3, \dots, N$ [7]. See also [8] for a detailed description of W -symmetry in conformal field theory. Their large N 't Hooft limit is defined as

$$N, k \rightarrow \infty, \quad \lambda \equiv \frac{N}{k+N} \quad \text{fixed}, \quad (1.1)$$

where 't Hooft coupling λ is a function of N and $k(=1, 2, \dots)$ and runs from zero to 1. The k is the level of the WZW current algebra. The central charge is given by $c_N(\lambda) \simeq N(1 - \lambda^2)$ under (1.1). The bulk theory they found is a Vasiliev type higher spin theory [9, 10, 11] in three-dimensional AdS space coupled with two complex (equally massive) scalar fields where the mass of the fields is given by $M^2 = -(1 - \lambda^2)$ which lies between -1 and zero. The above two complex scalars are quantized with opposite (conformally invariant) boundary conditions. Therefore, their conformal dimensions are $h_+ = \frac{1}{2}(1 + \lambda)$ and $h_- = \frac{1}{2}(1 - \lambda)$. The check for this duality is based on two aspects. 1) The two partition functions are found to match. The total partition function in the bulk consists of the sum of the contributions from both higher spin fields and the two complex scalar fields. It is quite nontrivial to find the conformal field theory partition function from the character formula within the large (N, k) 't Hooft limit. 2) The renormalization group (RG) flow patterns are coincident with each other. The RG flow for large $N > 2$ in the boundary theory is assumed to be obtainable by following the RG flow for $N = 2$ and the AdS/CFT correspondence is used in the bulk theory by interpreting the RG flow as a change of boundary conditions on one of the fields.

It is natural to ask whether there exist any higher spin AdS_3 gravity duals to other types of unitary coset minimal models. Some time ago, Lukyanov and Fateev [12] classified other types of (extended) W -symmetry algebras: WD_n symmetry algebras generated by currents of spins

$$WD_n : 2, 4, \dots, 2(n-1), \text{ and } n \quad (1.2)$$

and WB_n symmetry algebra generated by currents of spins

$$WB_n : 2, 4, \dots, 2n, \text{ and } (n + \frac{1}{2}). \quad (1.3)$$

The conformal dimension and the spin are linear combinations of the holomorphic conformal dimension and its antiholomorphic counterpart. More precisely, the above currents (left

component) have spins which are holomorphic conformal dimensions. Of course, their counterparts (right component) have spins which are opposite to its antiholomorphic conformal dimensions. Sometimes the latter algebra is denoted as $WB(0, n)$ with a Lie superalgebra $B(0, n) = OSp(1, 2n)$ (for example, in [8]) because the spin contents (1.3) come from the results of the Drinfeld-Sokolov reduction to this superalgebra rather than B_n itself. In this paper, by following the procedure of [1], we describe the coset WZW theories based on the above minimal models described by (1.2) and (1.3). Mainly we focus on their behaviors under the large (N, k) 't Hooft limit (1.1) and once we have found the two-dimensional results from the RG flows, then we reconsider them in the bulk using the AdS/CFT correspondence.

In section 2, we consider the diagonal coset minimal $WD_n^{(p)}$ model, where p is a minimal model index. By reading off conformal dimensions for primary fields developed in [12, 13, 14], we compute conformal dimensions for the relevant primary field and the other nontrivial lowest primary field (which has a nontrivial operator product expansion with the relevant field) in the large N 't Hooft limit (1.1). For known fusion rules between these two primaries with explicit structure constants (or three-point functions), we analyze the RG flow (due to the presence of the above relevant field) between the two fixed points: One fixed point is described by the $WD_n^{(p)}$ minimal model and the other one by the $WD_n^{(p-1)}$ model in which the minimal model index is shifted by 1. The description of the bulk theory, a higher spin theory coupled to two equally massive 'real' scalar fields, is obtained from the AdS/CFT correspondence. We also describe the total partition function in the bulk/boundary.

In section 3, we describe the procedures of section 2 for the case of the $WB_n^{(p)}$ model [12, 15]. We only present the main results without the details.

In section 4, we summarize what we have presented in this paper and comment on some future research directions. In particular, we briefly sketch out the large (N, k) 't Hooft limit for the second parafermion theory found in [16]. Finally, we describe one of the possible supersymmetric versions of the proposal [1].

Another proposal for large N limits of two dimensional solvable conformal field theories with their AdS duals is found in [17].

2 The large (N, k) limit of coset minimal $WD_n^{(p)}$ model

Let us consider the 'diagonal' coset WZW model characterized by [12, 3]

$$\frac{\widehat{SO}(N)_k \oplus \widehat{SO}(N)_1}{\widehat{SO}(N)_{k+1}}. \quad (2.1)$$

Denoting the spin 1 current fields of the affine Lie algebra $\widehat{SO}(N) \oplus \widehat{SO}(N)$ as $E_{(1)}^{ab}(z)$ and $E_{(2)}^{ab}(z)$, of levels k and 1, respectively, and the spin 1 current field of the diagonal affine Lie subalgebra $\widehat{SO}(N)$, which has level $(k+1)$, as $E'^{ab}(z)$, we have the relation $E'^{ab}(z) = E_{(1)}^{ab}(z) + E_{(2)}^{ab}(z)$. The level of the diagonal subalgebra is the sum of the other two levels because $E_{(1)}^{ab}(z)$ and $E_{(2)}^{ab}(z)$ commute with each other. The coordinate z is the complex coordinate in two dimensional conformal field theory. The indices a, b take the values $a, b = 1, 2, \dots, N$ in the representation of finite-dimensional Lie algebra $SO(N)$. These current fields of the WZW model are antisymmetric in the indices a, b and satisfy the standard operator product expansion [18, 8]. We introduce a rank n for $D_n = SO(2n)$ with a relation

$$N \equiv 2n. \quad (2.2)$$

The coset Virasoro generator $\tilde{T}(z)$ in (2.1) can be constructed from the relation $\tilde{T}(z) = T_{(1)}(z) + T_{(2)}(z) - T'(z)$. The stress energy tensors can be obtained from the Sugawara construction [8]; they are quadratic in the currents. Of course, $\tilde{T}(z)$ commutes with the diagonal current $E'^{ab}(z)$, which can be shown by computing the operator product expansion between them (and similarly with $T'(z)$). The central charge of the coset Virasoro algebra is $\tilde{c} = c_{(1)} + c_{(2)} - c'$, which can be seen by computing the operator product expansion between $T_{(1)}(z) + T_{(2)}(z) - T'(z)$ and $T_{(1)}(w) + T_{(2)}(w)$ in which we use the fact that $\tilde{T}(z)$ commutes with $T'(w)$. The operator product expansion between $T'(z)$ and $T_{(1)}(w) + T_{(2)}(w) (= \tilde{T}(w) + T'(w))$ is equivalent to $T'(z)T'(w)$ and then the above operator product expansion is $T_{(1)}(z)T_{(1)}(w) + T_{(2)}(z)T_{(2)}(w) - T'(z)T'(w)$. The coset central charge is a sum of three parts. Then the coset central charge is a function of p (2.4) as follows [12]:

$$\begin{aligned} c_N(p) &= \frac{1}{2}N(N-1) \left[\frac{k}{k+(N-2)} + \frac{1}{1+(N-2)} - \frac{k+1}{k+1+(N-2)} \right] \\ &= \frac{N}{2} \left[1 - \frac{(N-2)(N-1)}{p(p+1)} \right] \leq \frac{N}{2}, \end{aligned} \quad (2.3)$$

where the parameter p is introduced as a function of N and level k indicating the minimal model index

$$p \equiv k + N - 2 \geq N - 1, \quad k = 1, 2, \dots. \quad (2.4)$$

We used in (2.3) the fact that the dual Coxeter number of $SO(N)$ is given by $h^\vee = N - 2$ and the dimension of $SO(N)$ is $\dim SO(N) = \frac{1}{2}N(N-1)$. As in the $A_{n-1}^{(p)}$ minimal model, the maximum value of the central charge is the rank of $SO(N)$. According to the construction of [12], the spin 2 stress energy tensor can be written in terms of n -component massless scalar

fields. The second order derivatives of these scalar fields have a background charge α . When this background charge satisfies $\alpha^2 = \frac{1}{p(p+1)}$, then the central charge $c = n - 6\alpha^2\bar{\rho}^2$ becomes (2.3) where the Weyl vector $\bar{\rho}$ will be given later in (2.5). Therefore, the quantum Drinfeld-Sokolov description [8] for the central charge is equivalent to the coset description above. For $N = 6$ (or $n = 3$), the conformal field theory of the $WD_3^{(p)}$ minimal model is discussed in [19].

Are any critical behaviors of known statistical systems included in this unitary minimal series (2.3) and (2.4)? When $p = N - 1$ (the lowest value of p), then (2.3) implies that the central charge is $c = 1$, which describes the particular case of the critical behavior of the Ashkin-Teller model [20]. For the next lowest value, $p = N$, the model can be reduced to the Z_{2N} Ising model [16]. When $p \rightarrow \infty$ (by taking $k \rightarrow \infty$ with fixed N), the central charge is given by $c = \frac{N}{2}$. In this case, the symmetry algebra is the Casimir algebra of $\widehat{SO}(N)$ at level 1. This can be realized in terms of N real independent free fermions [18] (or see the papers [21, 22] for similar considerations), each of which contributes $\frac{1}{2}$ to the central charge. The spin 1 current is quadratic in these fermions. Note that the contributions from $c_{(1)}$ and $-c'$ in the first term and third term of (2.3), in this limit, exactly cancel each other. Then only the second term from $c_{(2)}$ remains and leads to $c = \frac{N}{2}$. The $A_{n-1}^{(p)}$ minimal model is realized by $(N - 1)$ free bosons.

The primary operators of the minimal model we are interested in are represented by the vertex operators that can be associated with the weight lattice of D_n (or $D_{\frac{N}{2}}$ via (2.2)) [12]. The weight vector that appears as the exponent of the vertex operator is labelled by two weight lattices denoted by α_+ and α_- (which are two Coulomb gas parameters). The allowed values of this weight vector should satisfy the condition for ‘strongly’ degenerate modules with respect to the chiral algebra. Then the field theory can be constructed from a finite number of primary fields. By introducing $\vec{n} = (n_1, n_2, \dots, n_n) = \sum_{i=1}^n n_i \vec{w}_i$ to represent the α_+ side and $\vec{n}' = (n'_1, n'_2, \dots, n'_n) = \sum_{i=1}^n n'_i \vec{w}_i$ to represent the α_- side, where \vec{w}_i with $i = 1, 2, \dots, n$ are the fundamental weights of the algebra D_n , and writing the background charge in terms of the Weyl vector $\bar{\rho} = (1, 1, \dots, 1) = \sum_{i=1}^n \vec{w}_i$, it is known that the Coulomb gas formula for the conformal dimension $\Delta_{(\vec{n}|\vec{n}')}^{(p)}$ of the primary operator $\Phi_{(\vec{n}|\vec{n}')}^{(p)}$ can be summarized by [12, 13, 14]

$$\Delta_{(\vec{n}|\vec{n}')}^{(p)} = \frac{1}{4p(p+1)} \left[((p+1)\vec{n} - p\vec{n}')^2 - \bar{\rho}^2 \right], \quad \bar{\rho}^2 = \frac{1}{3}n(n-1)(2n-1). \quad (2.5)$$

The positive integers n_i and n'_i are ‘Dynkin labels’. For the standard notation of [23], one needs to subtract the components of Weyl vector from this Dynkin label. In order to compute the conformal dimension (2.5) for various $(\vec{n}|\vec{n}')$ explicitly, the quadratic form matrix (the metric tensor ² for the weight space) for D_n is used [14]. For example, the square of the Weyl

²For convenience, we present the products of the weights: $\vec{w}_i \cdot \vec{w}_j = 2i$ for $i \leq j < n-1$, $\vec{w}_i \cdot \vec{w}_{n-1} = \vec{w}_i \cdot \vec{w}_n = i$

vector, $\bar{\rho}^2$, appearing in (2.5) is the sum of the quadratic form matrix elements. There is a difference in the overall factor compared to [12, 13, 18]. We also follow the Dynkin label notation of [15] instead of using the notation of [14]. The α_+ and α_- are written in terms of a parameter p : $\alpha_+ = \sqrt{\frac{p+1}{p}}$ and $\alpha_- = -\sqrt{\frac{p}{p+1}}$. The positive integers n_i and n'_i should satisfy some conditions, i.e., each linear combination of n_i and n'_i is bounded by the minimal model index p . The primary fields $\Phi_{(\vec{n}|\vec{n}')}^{(p)}$ with dimensions given by (2.5) together with their descendants form a closed operator algebra. The character of the module [12] can be written as $\frac{1}{\eta(\tau)^n} \exp \left[2\pi i \tau \left(\Delta_{(\vec{n}|\vec{n}')}^{(p)} - \frac{c_N(p)-n}{24} \right) \right]$ where $\eta(\tau)$ is the Dedekind function and τ is the modular parameter. It is easy to check that the last term of (2.5) cancels the dimension-independent parts of the character and the remaining terms of (2.5) contribute to the final character. Note that the combination $\frac{1}{24}(c_N(p) - n)$ appears in the quantum Drinfeld-Sokolov construction [8] for the conformal dimension.

Let us consider the neighborhood of the critical point of the $D_n^{(p)}$ model (a minimal model of the main series labelled by p (2.4) associated with a simple Lie algebra D_n of rank n) with p very much larger than n . The perturbed action, with a slightly different notation for the primary field, is given by Fateev and Lukyanov [12]

$$S_g^{(p)} = S_0^{(p)} + g \int d^2x \Phi_{(1^n|1,2,1^{n-2})}^{(p)}(x), \quad (2.6)$$

where $S_0^{(p)}$ is the action of the conformal field theory of the unperturbed $D_n^{(p)}$ model. See also the original papers by Zamolodchikov [24, 25] for the details. We use a simplified notation for the vectors indicating the representations of D_n in weight space: $(1^n) \equiv (1, 1, \dots, 1)$ which is a trivial representation of D_n and $(1, 2, 1^{n-2}) \equiv (1, 2, 1, \dots, 1)$ which is an adjoint representation of D_n ³. The number of elements should be equal to n . In the notation of [23], the former is (0^n) and the latter is given by $(0, 1, 0^{n-2})$. Note that in [14] more general perturbations are considered. There are multiple relevant operators with slightly relevant terms quadratic in the energy operator. In order to obtain the perturbation (2.6) from the description of [14], one should take the appropriate limit.

One can easily check that the dimension of the identity operator, $\Delta_{(\vec{n}|\vec{n})}^{(p)}$, for the representation with $\vec{n} = \vec{n}' = (1^n)$ vanishes because the numerator of (2.5) is identically zero. From (2.5), one can write the conformal dimension, by expanding, recollecting terms, and taking

for $i < n-1$, $\vec{w}_n \cdot \vec{w}_n = \vec{w}_{n-1} \cdot \vec{w}_{n-1} = \frac{n}{2}$, and $\vec{w}_{n-1} \cdot \vec{w}_n = \frac{n-2}{2}$.

³For the $A_{n-1}^{(p)}$ minimal model considered in [1], the perturbed action [12] is given by $S_g^{(p)} = S_0^{(p)} + g \int d^2x \Phi_{(1^{n-1}|2,1^{n-3},2)}^{(p)}(x)$. The ‘Dynkin label’ $(2, 1^{n-3}, 2)$, which is equivalent to $(1, 0^{n-3}, 1)$ of [23], represents the adjoint representation of A_{n-1} .

the large p limit, as follows:

$$\Delta_{(\vec{n}|\vec{n}')}^{(p)} = \frac{1}{4}(\vec{n} - \vec{n}')^2 + \frac{1}{4}(\vec{n}^2 - \vec{n}'^2)\epsilon + \mathcal{O}(\epsilon^2), \quad \epsilon \equiv \frac{1}{p+1} \simeq \frac{1}{p}. \quad (2.7)$$

The matrix of scalar products of the fundamental weights of the Lie algebra D_n is assumed here (in footnote 2). There are infinitely many solutions for (2.7) to possess slightly relevant fields (which have the conformal dimension 1 approximately) as $p \rightarrow \infty$. However, for the choice of the trivial α_+ side with (1^n) , there is a unique relevant field as in (2.6) above because the $(2, 2)$ component of the quadratic form matrix (in footnote 2) is equal to 4 and this provides a constant term 1 in (2.7)⁴. More explicitly, one can compute the conformal dimension for the relevant field (adjoint representation) from (2.5) as follows:

$$\Delta_{(1^n|1,2,1^{n-2})}^{(p)} = \frac{(p - N + 3)}{(p + 1)} \simeq 1 - \lambda, \quad \lambda \equiv \frac{N}{k + N}, \quad (2.8)$$

where we take the large (N, k) 't Hooft limit with fixed 't Hooft coupling λ defined as (1.1) of [1] in the last line of (2.8) here. In the context of [26] where perturbation by an appropriate operator leads to an IR fixed point described by the coset $\frac{\widehat{SO(N)}_{k-1} \oplus \widehat{SO(N)}_1}{\widehat{SO(N)}_k}$ (this can be obtained from (2.1) by replacing k with $k - 1$), one can understand that the conformal dimension for an appropriate field is given by the dual Coxeter number and the levels to be $\Delta_{(1^n|1,2,1^{n-2})}^{(p)} = 1 - \frac{h^\nu}{k+1+h^\nu} = 1 - \frac{N-2}{p+1}$ which is exactly the same as (2.8). Of course, this description for the $A_{n-1}^{(p)}$ minimal model can be analyzed and it can be seen that the behavior of (2.8) has features in common with the conformal dimension of the adjoint representation of A_{n-1} . It is not obvious how one can obtain the conformal dimensions from the coset model itself [27]. See the papers [28] or [1] for the explicit formula. From the quadratic Casimir $(N - 2)$ for the adjoint representation of $SO(N)$, one can write down the conformal dimension as $1 - \frac{(N-2)}{(N-2)+k+1} = 1 - \frac{N-2}{p+1}$ which is exactly the same as (2.8). That is, the first and second representations in the coset model (2.1) are trivial representations of $SO(N)$ while the diagonal representation is the adjoint representation of $SO(N)$. Here the quadratic Casimir is defined as $\frac{1}{4}(\vec{n}^2 - \vec{\rho}^2)$ for the representation \vec{n} of $SO(N)$ and we will use this formula in the remaining parts of this paper.

We noticed that the identity operator has a conformal dimension of zero. What is the lowest dimension operator, after the identity operator, in the singlet sector? What happens

⁴More precisely, there exists a unique 'slightly' relevant field. A relevant field, in general, has conformal dimension less than 1 ($\Delta < 1$) because the scaling dimension should be less than 2 which is the dimension of conformal field theory. In this case the scaling dimension with no spin is the sum of the holomorphic conformal dimension (Δ) and its antiholomorphic counterpart ($\overline{\Delta}$). That is, $\Delta + \overline{\Delta} = 2\Delta < 2$. For example, the primary field $\Phi_{(1^n|2,1^{n-1})}^{(p)}$ is also a relevant field because its conformal dimension is less than 1 due to (2.9). However, this relevant field is a 'strongly' relevant field and so one cannot analyze perturbatively. On the other hand, perturbative analysis is possible for the 'slightly' relevant field which has conformal dimension close to 1.

if we take $(2, 1^{n-1})$, which is a defining representation of D_n , as the α_- side as well as the trivial α_+ side (1^n) ? One computes the conformal dimension for the primary field $\Phi_{(1^n|2, 1^{n-1})}^{(p)}$ exactly and takes the large (N, k) 't Hooft limit as before to obtain

$$\Delta_{(1^n|2, 1^{n-1})}^{(p)} = \frac{(p - N + 2)}{2(p + 1)} \simeq \frac{1}{2}(1 - \lambda). \quad (2.9)$$

This primary field is identified with the energy operator in [14]. Note that the factor $\frac{1}{2}$ comes from the $(1, 1)$ component of the quadratic form matrix which is equal to 2 (see footnote 2), together with the overall factor $\frac{1}{4}$ in the formula (2.5). Obviously at finite (N, k) , this expression is different from that of the fundamental representation of the $A_{(n-1)}^{(p)}$ minimal model. However, they have common behavior in the large (N, k) 't Hooft limit. Furthermore, one can compute the conformal dimension $\Delta_{(1^n|1^{n-1}, 2)}^{(p)}$ when the integer 2 arises as the last Dynkin label rather than the first label as in (2.9). From the relation (2.7), the constant piece looks like the (n, n) -component of the quadratic form matrix, which is equal to $\frac{N}{4}$. This is rather different to the $A_{n-1}^{(p)}$ minimal model where the corresponding dimension behaves as $\frac{N-1}{N}$. From the quadratic Casimir $\frac{1}{2}(N-1)$ for the defining representation in $SO(N)$, one can write down the conformal dimension as $\frac{1}{2}(N-1)[\frac{1}{(N-2)+1} - \frac{1}{(N-2)+k+1}] = \frac{1}{2} - \frac{N-1}{2(p+1)}$ which is exactly the same as (2.9) where we used the quadratic Casimir $(N-2)$ for the adjoint representation in the denominator. The first representation of (2.1) is a trivial representation of $SO(N)$.

The operator product expansions of the fields $\Phi_{(\vec{n}|\vec{n}')}^{(p)}$ and $\Phi_{(\vec{m}|\vec{m}')}^{(p)}$ are, in general, linear combinations of $\Phi_{(\vec{s}|\vec{s}')}^{(p)}$ with the appropriate structure constants of the operator algebra. The selection rules of the operator algebra may be described by the Clebsch-Gordan series for the product of the finite-dimensional representations of the Lie algebra D_n with highest weights specified by the sets of the numbers (n_i, n'_i) and (m_i, m'_i) corresponding to the weight vectors. Although the structure constants are determined by three-point correlation functions through the Coulomb gas formalism, it is a rather nontrivial task to find them explicitly. Luckily, the four necessary integrals from the Coulomb gas formalism have been computed and the structure constants are written in terms of these integrals. Eventually, the fusion rules between the two primaries (adjoint and defining representations) described by (2.8) and (2.9) can be summarized by [14]

$$\begin{aligned} \Phi_{(1^n|2, 1^{n-1})}^{(p)} \otimes \Phi_{(1^n|2, 1^{n-1})}^{(p)} &= \Phi_{(1^n|1, 2, 1^{n-2})}^{(p)} + \cdots, \\ \Phi_{(1^n|1, 2, 1^{n-2})}^{(p)} \otimes \Phi_{(1^n|1, 2, 1^{n-2})}^{(p)} &= \Phi_{(1^n|1, 2, 1^{n-2})}^{(p)} + \cdots, \\ \Phi_{(1^n|1, 2, 1^{n-2})}^{(p)} \otimes \Phi_{(1^n|2, 1^{n-1})}^{(p)} &= \Phi_{(1^n|2, 1^{n-1})}^{(p)} + \cdots, \end{aligned} \quad (2.10)$$

where we have ignored the identity operator and the terms on the right hand side that are

irrelevant (in the context of RG analysis). The structure constants appearing in the right hand side are obtained from the three-point correlation functions of the unperturbed $D_n^{(p)}$ model [14]. When we look at the operator product expansion between $\Phi_{(1^n|2,1^{n-1})}^{(p)}(z)$ and $\Phi_{(1^n|2,1^{n-1})}^{(p)}(w)$, there exists a factor $(z-w)^{-4\Delta_{(1^n|2,1^{n-1})}^{(p)}+2\Delta_{(1^n|1,2,1^{n-2})}^{(p)}}$ in the right hand side of the first equation of (2.10). Substituting the conformal dimensions (2.8) and (2.9) into this exponent, gives the factor $(z-w)^{\frac{6}{p+1}}$ which goes to 1 in the large p limit. Then, the normal ordered field product [8] (the constant term in the operator product expansion) of $\Phi_{(1^n|2,1^{n-1})}^{(p)}(z)$ and $\Phi_{(1^n|2,1^{n-1})}^{(p)}(z)$, denoted by $(\Phi_{(1^n|2,1^{n-1})}^{(p)}\Phi_{(1^n|2,1^{n-1})}^{(p)})(z)$, is given by $\Phi_{(1^n|1,2,1^{n-2})}^{(p)}(z)$ up to the structure constant which is equal to $\sqrt{2}$ for large N , as follows

$$(\Phi_{(1^n|2,1^{n-1})}^{(p)}\Phi_{(1^n|2,1^{n-1})}^{(p)})(z) \simeq \Phi_{(1^n|1,2,1^{n-2})}^{(p)}(z). \quad (2.11)$$

In other words, in the large (N, k) 't Hooft limit, the conformal dimension (2.8) of the perturbing primary field (adjoint representation) is equal to twice the conformal dimension (2.9) of the primary field (defining representation). This is a new feature under the large (N, k) 't Hooft limit. For the $A_{n-1}^{(p)}$ minimal model, the normal ordered field product between the fundamental representation and the anti-fundamental representation of A_{n-1} is the (perturbing) adjoint representation of A_{n-1} in the large (N, k) 't Hooft limit [1].

There exists a new critical point corresponding to the zero of the β -function at nonzero g [12, 14]. Due to the decrease of the c -function along the RG flow, this new critical point should correspond to the critical behavior of the $D_n^{(p')}$ model with $p' < p$ [24]. Note the p -dependence of the central charge (2.3). How does one determine p' in the RG analysis? The central charge at this new critical point can be determined by substituting $g^* = 4(n-1)\frac{\epsilon}{C} + \mathcal{O}(\epsilon^2)$ (which is the solution of the β -function, where the ϵ here is the same as the one in (2.7)) into the expression for the central charge $c_N(p)$ expanded in g [12], together with (2.3) and (2.7). It is found to be

$$c_N(p)^* = c_N(p) - \frac{64(n-1)^3\epsilon^3}{C^2} = c_N(p) - \frac{4n(n-1)(2n-1)}{p^3} \simeq c_N(p-1), \quad (2.12)$$

where C is the structure constant appearing in front of $\Phi_{(1^n|1,2,1^{n-2})}^{(p)}(w)$ on the right hand side of the operator product expansion (2.10) between $\Phi_{(1^n|1,2,1^{n-2})}^{(p)}(z)$ and $\Phi_{(1^n|1,2,1^{n-2})}^{(p)}(w)$ and it is given by [12, 14]

$$C_{(1^n|1,2,1^{n-2})(1^n|1,2,1^{n-2})}^{(1^n|1,2,1^{n-2})} = \frac{4(n-1)}{\sqrt{n(2n-1)}} + \mathcal{O}(\epsilon). \quad (2.13)$$

The correction term in (2.12) comes from $-12(n-1)\epsilon g^2 + 2Cg^3 + \dots$ at $g = g^*$. The field theory, given by (2.6) which has the UV behavior described by the $D_n^{(p)}$ model, at $g > 0$, has

also IR asymptotic behavior that is described by the $D_n^{(p-1)}$ model ⁵. The equation (2.12) implies that the central charge at a nonzero fixed point agrees with that of the $D_n^{(p-1)}$ model. The perturbation of the coset theory by an appropriate operator $\Phi_{(1^n|1,2,1^{n-2})}^{(p)}$ changes p into $(p-1) = p'$ where the difference 1 is nothing but the shift parameter (the level of the second spin 1 current $E_{(2)}^{ab}(z)$ of D_n) of the coset (2.1). In the large (N, k) 't Hooft limit, the RG flow changes the 't Hooft coupling, from p to $p-1$ (or k to $k-1$), as $\delta\lambda = \frac{\lambda^2}{N}$ and this implies that $\delta c = -N\lambda\delta\lambda = -\lambda^3$ which can be seen from (2.12). We used the fact that $c_N(\lambda) \simeq \frac{N}{2}(1-\lambda^2)$.

In order to understand the IR behaviors of the primary fields, one should consider the case where the α_- side is given by the trivial representation (1^n) . That is, when the α_+ side and α_- side for the weight vector are interchanged in (2.8) and (2.9), one can compute the following dimensions for the defining representation and an adjoint representation of D_n exactly, as well as its large (N, k) 't Hooft limit, using the conformal dimension formula (2.5) in order to see how the primaries corresponding to (2.8) and (2.9) flow along the RG,

$$\begin{aligned}\Delta_{(2,1^{n-1}|1^n)}^{(p)} &= \frac{(p+N-1)}{2p} \simeq \frac{1}{2}(1+\lambda), \\ \Delta_{(1,2,1^{n-2}|1^n)}^{(p)} &= \frac{(p+N-2)}{p} \simeq 1+\lambda.\end{aligned}\tag{2.14}$$

Note that the sum of (2.9) and the first equation of (2.14), for the defining representation, is equal to 1 under the large (N, k) 't Hooft limit. That is $\Delta_{(1^n|2,1^{n-1})}^{(p)} + \Delta_{(2,1^{n-1}|1^n)}^{(p)} \simeq 1$. Similarly, $\Delta_{(1^n|1,2,1^{n-2})}^{(p)} + \Delta_{(1,2,1^{n-2}|1^n)}^{(p)} \simeq 2$. The behavior of (2.14) in the large (N, k) 't Hooft limit is the same as those in the $A_{n-1}^{(p)}$ minimal model. From the quadratic Casimir $\frac{1}{2}(N-1)$ for the defining representation and quadratic Casimir $(N-2)$ for the adjoint representation in $SO(N)$, one can write down the conformal dimensions, in the coset model directly, as $\frac{1}{2}(N-1)[\frac{1}{(N-2)+k} + \frac{1}{(N-2)+1}] = \frac{1}{2} + \frac{N-1}{2p}$ and $1 + \frac{(N-2)}{(N-2)+k} = 1 + \frac{N-2}{p}$. These coincide with (2.14) as we expected. In the former, the diagonal representation is a trivial representation and in the latter, both the second and diagonal representations are trivial ones. In all cases we use the formula for the quadratic Casimir that was given earlier.

The slope of the β -function at the fixed point [29] provides the conformal dimension at the IR fixed point via $\frac{d\beta}{dg}|_{g^*} = -2(n-1)\epsilon + \mathcal{O}(\epsilon^2)$. Then the anomalous dimension for the relevant field (adjoint representation) at the IR fixed point is given by $\Delta \simeq 1 + \frac{2(n-1)}{p+1} \simeq 1 + \frac{N}{p-1}$. This is exactly the conformal dimension (2.14) of $\Phi_{(1,2,1^{n-2}|1^n)}^{(p-1)}$ in $D_n^{(p-1)}$ minimal model and

⁵ For $A_{n-1}^{(p)}$ minimal model, one can analyze similarly and the central charge is $c_N(p)^* = c_N(p) - \frac{8n^3\epsilon^3}{C^2} = c_N(p) - \frac{2n(n^2-1)}{p^3} \simeq c_N(p-1)$ where C is given by the result of the three-point correlation function at leading order to be $C_{(1^{n-1}|2,1^{n-3},2)(1^{n-1}|2,1^{n-3},2)}^{(1^{n-1}|2,1^{n-3},2)} = \frac{2n}{\sqrt{n^2-1}} + \mathcal{O}(\epsilon)$ [12].

therefore this leads to the flow

$$\text{UV} : \Phi_{(1^n|1,2,1^{n-2})}^{(p)}(z) \longrightarrow \text{IR} : \Phi_{(1,2,1^{n-2}|1^n)}^{(p-1)}(z). \quad (2.15)$$

We also get a similar relation to (2.11) in the IR region for the $D_n^{(p-1)}$ model from the same analysis that was done in (2.11)

$$(\Phi_{(2,1^{n-1}|1^n)}^{(p-1)} \Phi_{(2,1^{n-1}|1^n)}^{(p-1)})(z) \simeq \Phi_{(1,2,1^{n-2}|1^n)}^{(p-1)}(z). \quad (2.16)$$

It is easy to see from (2.15) how the flow of the primary field of the defining representation of D_n arises along the RG flow by realizing that the left hand side of (2.15) is given by the product of two defining representations via (2.11) and the right hand side of (2.15) is given by the product of other defining representations via (2.16).

Alternatively, one can directly obtain the flow of the primary field of the defining representation. From (2.14) and (2.9), one also obtains

$$\Delta_{(2,1^{n-1}|1^n)}^{(p-1)} - \Delta_{(1^n|2,1^{n-1})}^{(p)} = \left[\frac{1}{2} + \frac{N-1}{2(p-1)} \right] - \left[\frac{1}{2} - \frac{N-1}{2(p+1)} \right] \simeq \lambda. \quad (2.17)$$

On the other hand, the observation of Cardy and Ludwig [30] implies that the correction to the conformal dimension for small deviations from the new fixed point is given by three quantities: two structure constants and the small parameter (which is related to our minimal series index p) of the theory. It is easy to check that

$$\sqrt{\frac{2n-1}{n}} \left(\frac{4(n-1)}{\sqrt{n(2n-1)}} \right)^{-1} 4(n-1)\epsilon = (2n-1)\epsilon \simeq \lambda, \quad (2.18)$$

where we used the result of [14] for the structure constant appearing in the operator product expansion between $\Phi_{(1^n|2,1^{n-1})}^{(p)}(z)$ and $\Phi_{(1^n|2,1^{n-1})}^{(p)}(w)$ in the first equation of (2.10) which is equal to $C_{(1^n|2,1^{n-1})(1^n|2,1^{n-1})}^{(1^n|1,2,1^{n-2})} = \sqrt{\frac{2n-1}{n}} + \mathcal{O}(\epsilon)$ and another structure constant given in (2.13). The last factor $4(n-1)\epsilon$ in (2.18) comes from the correction term of the central charge (2.12). By comparing (2.17) with (2.18), in the IR, the field $\Phi_{(1^n|2,1^{n-1})}^{(p)}$ of the $D_n^{(p)}$ minimal model is identified with the field $\Phi_{(2,1^{n-1}|1^n)}^{(p-1)}$ of the $D_n^{(p-1)}$ minimal model and therefore one sees the flow

$$\text{UV} : \Phi_{(1^n|2,1^{n-1})}^{(p)} \longrightarrow \text{IR} : \Phi_{(2,1^{n-1}|1^n)}^{(p-1)} \quad (2.19)$$

which is consistent with (2.11) and (2.16) in the fact that under the flow (2.19), the flow (2.15) is satisfied as we mentioned before. For the $A_{n-1}^{(p)}$ minimal model, one can perform a similar analysis and the computation of (2.17) gives $\frac{N^2-1}{2N}(\frac{1}{p-1} + \frac{1}{p+1}) \simeq \lambda$. Although we do not know

the structure constant between the two primary fields of the fundamental representations leading to the primary field of the adjoint representation (more precisely the coefficient of three-point function for these three fields), from the considerations of (2.17) and (2.19), one concludes with the help of footnote 5 that the large (N, k) 't Hooft limit for this unknown coefficient of the three-point function should be equal to 1. The analysis of the three-point function between the two primaries of the antifundamental representations and the primary of the adjoint representation can be done similarly. The basic generating fields, from which we produce all the states in the conformal field theory by taking the fusion products of them, are given by the following defining representations

$$\Phi_{(1^n|2,1^{n-1})}^{(p)} \quad \text{and} \quad \Phi_{(2,1^{n-1}|1^n)}^{(p)}. \quad (2.20)$$

From the operator product expansion in the first equation of (2.10), one can think of the irrelevant fields having the next lowest conformal dimension. From the Clebsh-Gordan coefficient between the two defining representations of $SO(N)$, one obtains the conformal dimensions of the primary field $\Phi_{(1^n|3,1^{n-1})}^{(p)}$, where the n'_1 component in \vec{n}' is greater than 1. Then one obtains the conformal dimension by using the formula (2.5) and moreover one can compute the conformal dimension for the other primary field $\Phi_{(3,1^{n-1}|1^n)}^{(p)}$ as follows:

$$\begin{aligned} \Delta_{(1^n|3,1^{n-1})}^{(p)} &= \frac{(2p - N + 2)}{(p + 1)} \simeq 2 - \lambda, \\ \Delta_{(3,1^{n-1}|1^n)}^{(p)} &= \frac{(2p + N)}{p} \simeq 2 + \lambda. \end{aligned} \quad (2.21)$$

In this case, the quadratic Casimir for the $(3, 1^{n-1})$ representation of $SO(N)$ is equal to N . So the coefficient of the N -term in the first equation of (2.21) originates from $N[-\frac{1}{(N-2)+k+1}] = -\frac{N}{p+1}$ while the coefficient of the N -term in the second equation comes from $N[\frac{1}{(N-2)+k}] = \frac{N}{p}$.

How does one understand the primary field $\Phi_{(1^n|3,1^{n-1})}^{(p)}$ which has the conformal dimension given in the first relation of (2.21)? The one-loop contribution from the real scalar field in the bulk is given by $Z_{\text{scal}}(h_-) = \prod_{l,l'=0}^{\infty} \frac{1}{(1-q^{h_-+l}\bar{q}^{h_-+l'})}$ where $h_- = \frac{1}{2}(1 - \lambda)$ and $q \equiv e^{2\pi i\tau}$. Here τ is the modular parameter which is the ratio of two complex periods of the lattice on a torus [18]. Expanding out the first few terms in $Z_{\text{scal}}(h_-)$, one has a $q^{h_-}\bar{q}^{h_-}$ term, a $q^{2h_-}\bar{q}^{2h_-}$ term and a $q^{2h_-+1}\bar{q}^{2h_-+1}$ term and so on. Since the conformal dimension for the adjoint representation is given by $\Delta_{(1^n|1,2,1^{n-2})}^{(p)} = 2\Delta_{(1^n|2,1^{n-1})}^{(p)} = 2h_-$ in the large (N, k) 't Hooft limit, eventually the terms with an overall factor $q^{2h_-}\bar{q}^{2h_-}$ should correspond to the character for the adjoint representation $(1^n|1, 2, 1^{n-2})$ in the total partition function. Here we should add the contribution Z_{hs} (the explicit form will be given later) from the gravitons of the higher spin fields. Similarly, the conformal dimension for the above irrelevant field is given

by $\Delta_{(1^n|3,1^{n-1})}^{(p)} = 2\Delta_{(1^n|2,1^{n-1})}^{(p)} + 1 = 2h_- + 1$ in the large (N, k) 't Hooft limit and the terms with an overall factor $q^{2h_-+1}\bar{q}^{2h_-+1}$ should correspond to the character for the representation $(1^n|3, 1^{n-1})$ in the total partition function which contains Z_{hs} .

Note that for the $A_{n-1}^{(p)}$ minimal model, the adjoint representation appears in the fusion product of fundamental and antifundamental representations and the fusion product of two fundamental representations give other representations. However, in the $D_n^{(p)}$ minimal model of this paper, the adjoint representation arises from the fusion product of two defining representations. The one-loop contribution from the other real scalar field in the bulk is given by $Z_{\text{scal}}(h_+) = \prod_{l,l'=0}^{\infty} \frac{1}{(1-q^{h_++l}\bar{q}^{h_++l'})}$ where $h_+ = \frac{1}{2}(1+\lambda)$. Expanding out the first few terms, one obtains a $q^{h_+}\bar{q}^{h_+}$ term, a $q^{2h_+}\bar{q}^{2h_+}$ term and a $q^{2h_++1}\bar{q}^{2h_++1}$ term. Since the conformal dimension for the adjoint representation is given by $\Delta_{(1,2,1^{n-2}|1^n)}^{(p)} = 2\Delta_{(2,1^{n-1}|1^n)}^{(p)} = 2h_+$ in the large (N, k) 't Hooft limit, the terms with an overall factor $q^{2h_+}\bar{q}^{2h_+}$ should correspond to the character for the adjoint representation $(1, 2, 1^{n-2}|1^n)$ in the total partition function. Similarly, the conformal dimension for the above irrelevant field is $\Delta_{(3,1^{n-1}|1^n)}^{(p)} = 2\Delta_{(2,1^{n-1}|1^n)}^{(p)} + 1 = 2h_+ + 1$ in the large (N, k) 't Hooft limit and the terms with an overall factor $q^{2h_++1}\bar{q}^{2h_++1}$ should correspond to the character for the representation $(3, 1^{n-1}|1^n)$ in the total partition function where the contribution from Z_{hs} should be added.

For the fusion product $\Phi_{(1^n|2,1^{n-1})}^{(p)} \otimes \Phi_{(2,1^{n-1}|1^n)}^{(p)} = \Phi_{(2,1^{n-1}|2,1^{n-1})}^{(p)}$ from different types of combinations in (2.20), one can compute the conformal dimension for the primary field appearing in the right hand side and see that it is given by $\Delta_{(2,1^{n-1}|2,1^{n-1})}^{(p)} = \frac{(N-1)}{2p(p+1)} \simeq \frac{\lambda^2}{2N}$. This is consistent with the computation from the coset model $\frac{1}{2}(N-1)[\frac{1}{(N-2)+k} - \frac{1}{(N-2)+k+1}]$ with the quadratic Casimir $\frac{1}{2}(N-1)$ for the $(2, 1^{n-1})$ representation of $SO(N)$ as before. The second representation of the coset is a trivial one. This is equal to the nonconstant piece on the left hand side of fusion rule. In other words, we have $\Delta_{(1^n|2,1^{n-1})}^{(p)} + \Delta_{(2,1^{n-1}|1^n)}^{(p)} = 1 + \frac{(N-1)}{2p(p+1)}$.

What is the AdS_3 dual gravity theory of the two-dimensional coset minimal model? The primary field $\Phi_{(1^n|1,2,1^{n-2})}^{(p)}(z)$ is the normal ordered product of $(\Phi_{(1^n|2,1^{n-1})}^{(p)}\Phi_{(1^n|2,1^{n-1})}^{(p)})(z)$ in (2.11) and the perturbation can be rewritten

$$g \int d^2x (\Phi_{(1^n|2,1^{n-1})}^{(p)}\Phi_{(1^n|2,1^{n-1})}^{(p)})(x) = g \int d^2x (\mathcal{O}\mathcal{O})(x), \quad (2.22)$$

where the primary field $\mathcal{O}(z) \equiv \Phi_{(1^n|2,1^{n-1})}^{(p)}(z)$ has holomorphic conformal dimension $\frac{1}{2}(1-\lambda)$ (2.9) in the large (N, k) 't Hooft limit. Its antiholomorphic conformal dimension is also $\frac{1}{2}(1-\lambda)$. In the AdS_3 gravity theory side from the AdS/CFT correspondence [31], the scalar field, corresponding to $\mathcal{O}(z)$, with dimension Δ_- (which is the sum of holomorphic and antiholomorphic conformal dimensions) is quantized in the $(-)$ quantization in the UV (see also the relevant paper [32]). In other words, the scalar field behaves as $\phi \sim r^{1-\lambda}$ with an

appropriate boundary condition where r is a radial coordinate in AdS_3 space. There exists an alternative choice for the quantization with an irrelevant perturbation by an operator of dimension $2 - (1 - \lambda) = (1 + \lambda)$, where ϕ' behaves as $r^{1+\lambda}$, but this is not the case in (2.22). Along the RG flow, this scalar field ϕ flows to the theory with (+) quantization in the IR where it corresponds to an operator $\mathcal{O}'(z) \equiv \Phi_{(2,1^{n-1}|1^n)}^{(p-1)}(z)$ with dimension $\frac{1}{2}(1 + \lambda)$ in the large (N, k) 't Hooft limit. The $(\mathcal{O}\mathcal{O})(z)$ in (2.22) flows to an irrelevant operator of the form $(\mathcal{O}'\mathcal{O})(z)$. The two solutions for the mass formula of matter multiplet $M^2 = \Delta(\Delta - 2)$ in higher spin theory are written as, by summing over holomorphic and antiholomorphic parts,

$$\Delta_- = \frac{1}{2}(1 - \lambda) + \frac{1}{2}(1 - \lambda) = 1 - \lambda, \quad \Delta_+ = \frac{1}{2}(1 + \lambda) + \frac{1}{2}(1 + \lambda) = 1 + \lambda. \quad (2.23)$$

Therefore, the two real scalar fields (ϕ, ϕ') in the AdS_3 gravity theory with $M^2 = -(1 - \lambda^2)$ where ϕ is in the $(-)$ quantization and ϕ' in the $(+)$ quantization match with the results for the RG flow in the two-dimensional dual conformal field theories we have described so far. Note that from (2.23) we have a relation $\Delta_- + \Delta_+ = 2$ or $\Delta_{\mathcal{O}} + \Delta_{\mathcal{O}'} = 1$ from (2.9) and (2.14). See also the relevant work [33] for the changing of conformal dimension $(1 - \lambda)$ into $(1 + \lambda)$ in the different context of gravitational dressing (see also [34]).

By following the procedure [35] for the one-loop determinant in the heat kernel techniques, one expects that the total one-loop determinant is given by the multiple product of each contribution for spin s . Then this can be interpreted using the boundary theory. The vacuum character for the simply laced algebra with level 1 is given by [8, 36]

$$\chi = \frac{1}{\prod_{i=1}^n F_{e_i+1}(q)}, \quad F_s(q) \equiv \prod_{k=s}^{\infty} (1 - q^k), \quad q \equiv e^{2\pi i \tau}. \quad (2.24)$$

This is the vacuum character of type $\mathcal{W}(e_1 + 1, e_2 + 1, \dots, e_n + 1)$ algebra in the notation of [8]. For the $A_{n-1}^{(p)}$ minimal model, the algebra consists of a spin 2 Virasoro generator and additional primary currents of spins $3, 4, \dots, n (= N)$. Now let us apply the $SO(N)$ group to (2.24) and realize that there exist n exponents of $SO(N)$: $e_1 = 1, e_2 = 3, \dots, e_{n-1} = 2n - 3$ and $e_n = n - 1$. By taking into account the antiholomorphic part, the large N limit of (2.24) can be written as

$$Z_{\text{hs}} = \lim_{N \rightarrow \infty} \left(\prod_{m=2}^{\infty} \frac{1}{|1 - q^m|^2} \prod_{m=4}^{\infty} \frac{1}{|1 - q^m|^2} \cdots \prod_{m=N-2}^{\infty} \frac{1}{|1 - q^m|^2} \prod_{m=\frac{N}{2}}^{\infty} \frac{1}{|1 - q^m|^2} \right). \quad (2.25)$$

This partition function from the $D_n^{(p)}$ minimal model conformal field theory should agree with that from the one-loop result in the higher spin bulk theory. Moreover, the higher spin theory we are interested in has two real scalar fields. The one-loop contributions from each scalar

field can be obtained from [37]. We also present the successive fusion products in the context of the conformal field theory partition function. The identifications,

$$\begin{aligned} Z_{\text{scal}}(h_-) &= \prod_{l,l'=0}^{\infty} \frac{1}{(1 - q^{h_-+l} \bar{q}^{h_-+l'})} \quad \leftrightarrow \quad \Phi_{(1^n|2,1^{n-1})}^{(p)} \otimes \cdots \otimes \Phi_{(1^n|2,1^{n-1})}^{(p)}, \\ Z_{\text{scal}}(h_+) &= \prod_{l,l'=0}^{\infty} \frac{1}{(1 - q^{h_++l} \bar{q}^{h_++l'})} \quad \leftrightarrow \quad \Phi_{(2,1^{n-1}|1^n)}^{(p)} \otimes \cdots \otimes \Phi_{(2,1^{n-1}|1^n)}^{(p)}, \end{aligned} \quad (2.26)$$

where $h_- \equiv \frac{1}{2}\Delta_-$ and $h_+ \equiv \frac{1}{2}\Delta_+$ imply that the left hand side of the first equation in (2.26) provides the contributions to the fusion product that contain the multiple copies of $\Phi_{(1^n|2,1^{n-1})}^{(p)}$ by extending the simplest product to the more general case. On the other hand, the left hand side of the second equation of (2.26) corresponds to the multiple copies of $\Phi_{(2,1^{n-1}|1^n)}^{(p)}$. Then the total partition function can be written in terms of the partition functions in (2.25) and (2.26) as

$$Z_{\text{tot}} = (q\bar{q})^{-\frac{c}{24}} Z_{\text{hs}} Z_{\text{scal}}(h_-) Z_{\text{scal}}(h_+). \quad (2.27)$$

In order to see the one-to-one correspondence precisely, the computation for Z_{hs} (2.25) should also be done in the bulk to see whether it really coincides with (2.25), which was obtained from the computation in the boundary. Moreover, we described some identifications in (2.26) but we did not show explicitly how the characters in the boundary exactly match with $Z_{\text{scal}}(h_{\mp})$ (2.26) obtained from the bulk. According to [2], they have a conformal character [12, 8] and take the large (N, k) 't Hooft limit. The branching function contains the character of $U(\infty)$ and furthermore the scalar partition functions in (2.26) can be written in terms of the characters of the representations of $U(\infty)$ ⁶. From this observation for the $A_{n-1}^{(p)}$ minimal

⁶Just after this paper was released in the arXiv, the two relevant papers [38] and [39] appeared in the arXiv. The former is the published version of [2] in which the partition function of the WA_{N-1} minimal model was obtained. The latter deals with the partition function of the $WD_{\frac{N}{2}}$ minimal model. One of the main results of [38] is as follows. Due to the fact that certain states become null and decouple from correlation functions (and therefore have to be removed from the spectrum), the careful limiting procedure shows that the resulting states that survive exactly match the gravity prediction. The simplest example is given by the fusion product $\Phi_{(2,1^{n-1}|1^n)}^{(p)} \otimes \Phi_{(1^n|2,1^{n-1})}^{(p)}$ where both α_+ and α_- are nontrivial. The conformal dimensions are not additive. That is, $1 = \Delta_{(2,1^{n-1}|1^n)}^{(p)} + \Delta_{(1^n|2,1^{n-1})}^{(p)} \neq \Delta_{(2,1^{n-1}|2,1^{n-1})}^{(p)}$. However, their analysis shows that there exists a descendant state with the conformal dimension $\Delta_{(2,1^{n-1}|2,1^{n-1})}^{(p)} = \Delta_{(2,1^{n-1}|1^n)}^{(p)} + \Delta_{(1^n|2,1^{n-1})}^{(p)} = 1$ in the conformal field theory representation labeled by $(2, 1^{n-1}|2, 1^{n-1})$. This becomes the generating state of the representation and the state ψ and its descendants ρ and ξ in (2.20) of [38] match with the gravity results. The ω becomes null and the ω and its descendants then decouple from the correlation functions. In this computation, they considered the 'strict' infinite N limit where the sum of the number of boxes and antiboxes in the Young tableau has maximum value in the conformal field theory partition function.

What about the $WD_{\frac{N}{2}}$ minimal model case? The above feature is related to the cross terms in the product of $Z_{\text{scal}}(h_+)$ and $Z_{\text{scal}}(h_-)$ in (2.26). According to the result of [39], the character in the conformal field

model it may be that one can also write down the scalar partition functions in terms of a sum over the characters of representations of $SO(\infty)$ (or its more general group $O(\infty)$). In order to understand this clearly, it is useful to look at the expansion of characters developed in [40, 41]. The sum over the Weyl group elements and the sum over the lattice (generated by the simple roots of the Lie algebra D_n) in the character formula [12] should be related to the sum over the characters of representations of $SO(N)$ in the large N limit.

3 The large (N, k) limit of coset minimal $WB_n^{(p)}$ model

Let us consider the same ‘diagonal’ coset model (2.1) where a rank n for the non-simply laced algebra $B_n = SO(2n + 1)$ has a relation

$$N \equiv 2n + 1. \quad (3.1)$$

The central charge is given by (2.3) with the minimal model index (2.4). For $N = 5$, some coset theories with different choices of levels are described in [21, 22, 42]. The primary operators of the minimal model are represented by the vertex operators that can be associated with the weight lattice of B_n (or $B_{\frac{N-1}{2}}$ via (3.1)) [12]. The Coulomb gas formula for the conformal dimension of the primary operator $\Phi_{(\vec{n}|\vec{n}')}^{(p)}$ in the Neveu-Schwarz sector where $(n_n - n'_n)$ is even can be summarized by [12, 15, 43]

$$\Delta_{(\vec{n}|\vec{n}')}^{(p)} = \frac{1}{2p(p+1)} \left[((p+1)\vec{n} - p\vec{n}')^2 - \vec{\rho}^2 \right], \quad \vec{\rho}^2 = \frac{1}{12}n(2n-1)(2n+1). \quad (3.2)$$

For the Ramond-Ramond sector where $(n_n - n'_n)$ is odd, there is an extra $\frac{1}{16}$ factor in the above dimension formula. More explicitly, one can compute the conformal dimensions for the lowest dimensional field and the relevant field from (3.2) respectively as follows:

$$\begin{aligned} \Delta_{(1^n|2,1^{n-1})}^{(p)} &= \frac{(p-N+2)}{2(p+1)} \simeq \frac{1}{2}(1-\lambda), \\ \Delta_{(1^n|1,2,1^{n-2})}^{(p)} &= \frac{(p-N+3)}{(p+1)} \simeq 1-\lambda. \end{aligned} \quad (3.3)$$

theory partition function consists of a linear combination of the Schur functions on the trivial representation (1^n) , on the adjoint representation $(1, 2, 1^{n-2})$ and on the representation $(3, 1^{n-1})$. It turns out that the states corresponding to the Schur function on the trivial representation, which are the states generated from ω , become null and decouple from the correlation function. Of course, the states from ψ and its descendants corresponding to the Schur functions on the adjoint representation $(1, 2, 1^{n-2})$ and on the representation $(3, 1^{n-1})$ match the gravity prediction. The total conformal field theory partition function agrees with the bulk partition function.

Although the quadratic form matrix for the B_n group ⁷ is different from that of the D_n group (in footnote 2) and the expression for the conformal dimension (3.2) looks different from that (2.5) of D_n , the expressions (2.8) and (2.9) at finite N and k are coincident with (3.3).

In the original paper [12], the RG analysis was described for $A_{n-1}^{(p)}$ and $D_n^{(p)}$ models only but the $B_n^{(p)}$ model can also be analyzed in a similar way. For example, the perturbed action is the same as in (2.6). The normal ordered product (2.11) holds by taking the large (N, k) 't Hooft limit. Then the central charge at the new critical point can be determined by substituting $g^* = 2(2n-1)\frac{\epsilon}{C} + \mathcal{O}(\epsilon^2)$ into the expression of the central charge $c_N(p)$ expanded in g

$$c_N(p)^* = c_N(p) - \frac{8(2n-1)^3\epsilon^3}{C^2} = c_N(p) - \frac{2n(2n-1)(2n+1)}{p^3} \simeq c_N(p-1), \quad (3.4)$$

where C is the structure constant corresponding to (2.13) for the $B_n^{(p)}$ minimal model and is given by [15], via the three-point function in the Coulomb gas representation (that is, the fusion constant and the normalization of the vertex operator), to be

$$C_{(1^n|1,2,1^{n-2})}^{(1^n|1,2,1^{n-2})} = \frac{2(2n-1)}{\sqrt{n(2n+1)}} + \mathcal{O}(\epsilon). \quad (3.5)$$

Of course, the motivation of [15] is to describe the RG flows for the second parafermion theory which will be described in next section but, as a by-product, they also found this structure constant through the Coulomb gas representation with a three-point function. Similar analysis gives the flow (2.15) for the $B_n^{(p)}$ minimal model under the RG flow with (2.16). One obtains the following conformal dimensions, corresponding to (3.3) but with the α_+ side and the α_- side interchanged, which allow us to understand how the primary fields transform under the RG flow,

$$\begin{aligned} \Delta_{(2,1^{n-1}|1^n)}^{(p)} &= \frac{(p+N-1)}{2p} \simeq \frac{1}{2}(1+\lambda), \\ \Delta_{(1,2,1^{n-2}|1^n)}^{(p)} &= \frac{(p+N-2)}{p} \simeq 1+\lambda. \end{aligned} \quad (3.6)$$

These match the conformal dimensions (2.14) for the $D_n^{(p)}$ model meaning that the two models show the same behavior.

The correction of the conformal dimension for a small deviation from the new fixed point can be written as

$$\frac{2n}{\sqrt{n(2n+1)}} \left(\frac{2(2n-1)}{\sqrt{n(2n+1)}} \right)^{-1} 2(2n-1)\epsilon = 2n\epsilon \simeq \lambda, \quad (3.7)$$

⁷For convenience, we present the elements here: $\vec{w}_i \cdot \vec{w}_j = i$ for $i \leq j < n$, $\vec{w}_i \cdot \vec{w}_n = \frac{i}{2}$ for $i < n$ and $\vec{w}_n \cdot \vec{w}_n = \frac{n}{4}$.

where the structure constant $C_{(1^n|2,1^{n-1})(1^n|2,1^{n-1})}^{(1^n|1,2,1^{n-2})} = \frac{2n}{\sqrt{n(2n+1)}} + \mathcal{O}(\epsilon)$ was found in [15] and the structure constant (3.5) is used. The fusion rules (2.10) are also valid for this case. In addition, the factor $2(2n-1)\epsilon$ is consistent with the correction term for the central charge in (3.4). On the other hand, there is a difference between the conformal dimensions, which can be computed from (3.3) and (3.6) to be

$$\Delta_{(2,1^{n-1}|1^n)}^{(p-1)} - \Delta_{(1^n|2,1^{n-1})}^{(p)} = \left[\frac{1}{2} + \frac{N-1}{2(p-1)} \right] - \left[\frac{1}{2} - \frac{N-1}{2(p+1)} \right] \simeq \lambda. \quad (3.8)$$

By comparing (3.7) with (3.8), in the IR, the field $\Phi_{(1^n|2,1^{n-1})}^{(p)}$ of the $B_n^{(p)}$ minimal model is identified with the field $\Phi_{(2,1^{n-1}|1^n)}^{(p-1)}$ of the $B_n^{(p-1)}$ minimal model. The relation (2.19) also holds for the $B_n^{(p)}$ minimal model.

According to [8], the vacuum character for B_n with level 1 has an extra contribution from the fermionic $(n + \frac{1}{2})$ dimensional field projected onto the Z_2 even sector. Odd Z_2 parity is assigned to the currents of half odd integer spin and even Z_2 parity is assigned to the integer spin currents [8]. The singlet algebra is the bosonic projection of the type $\mathcal{W}(2, 4, \dots, 2n = N-1, n + \frac{1}{2} = \frac{N}{2})$. Then the large N limit of the partition function for the higher spin with field contents (1.3) is written as

$$\lim_{N \rightarrow \infty} \left(\prod_{m=2}^{\infty} \frac{1}{|1 - q^m|^2} \cdots \prod_{m=N-1}^{\infty} \frac{1}{|1 - q^m|^2} \times \left| \frac{1}{2} \left[\prod_{m=\frac{N}{2}}^{\infty} (1 + q^{m+\frac{1}{2}}) + \prod_{m=\frac{N}{2}}^{\infty} (1 - q^{m+\frac{1}{2}}) \right] \right|^2 \right), \quad (3.9)$$

where the last term in (3.9) is the vacuum character of the above fermion field projected onto the Z_2 even sector. This is very similar to the bosonic projection of the $\mathcal{N} = 1$ superconformal algebra, which can be realized as the WB_1 minimal model because the field contents from (1.3) are given by a spin 2 Virasoro generator and spin $\frac{3}{2}$ superpartner, of type $\mathcal{W}(2, 4, 6)$ [36]. See also [44] for the coset currents of spin $(n + \frac{1}{2})$ and representation theory. Finally, one obtains the total partition function (2.27) where the higher spin part Z_{hs} is given by (3.9) for the $B_n^{(p)}$ minimal model.

4 Conclusions and outlook

We described the dualities between the large (N, k) 't Hooft limits of the $WD_n^{(p)}$ and $WB_n^{(p)}$ coset minimal models and the higher spin theory on AdS_3 where two massive real scalars are added to the massless higher spin fields. We explained this duality by showing that the RG flows of the two-dimensional conformal field theories agree with the gravity analysis from the AdS/CFT correspondence.

So far, the level of the second group in the coset model is 1. What happens if the ‘shift parameter’ is greater than 1? We present two examples. The first example, from a series of unitary conformal field theories, is the second parafermion theory by Fateev and Zamolodchikov [16]. Note that the first parafermion theory is a single conformal field theory for given N . The diagonal coset model, denoted by $Z_N^{(2)}(p)$, for $N \geq 5$ is characterized by [45, 27]

$$\frac{\widehat{SO}(N)_k \oplus \widehat{SO}(N)_2}{\widehat{SO}(N)_{k+2}}. \quad (4.1)$$

The central charge for (4.1) is given by [16] as

$$c_N(p) = (N-1) \left[1 - \frac{N(N-2)}{p(p+2)} \right] \leq (N-1), \quad p \equiv k + N - 2 \geq N-1, \quad (4.2)$$

which can be seen by realizing that the correct level for the second group in this case is 2 rather than 1. In the large (N, k) ’t Hooft limit, this reduces to $c_N(p) \simeq N(1 - \lambda^2)$ which is twice that of previous examples. For $N = 2$, the theory is given by $c = 1$ free boson theory. For $N = 3$ parafermion theory, developed in [46], it is known that the coset is given by $\frac{\widehat{SU}(2)_{2k} \oplus \widehat{SU}(2)_4}{\widehat{SU}(2)_{2k+4}}$ where $\widehat{SO}(3)_k$ is identified with $\widehat{SU}(2)_{2k}$. The two slightly relevant perturbations on this coset model are described in [47, 48] and there exists only a single IR fixed point denoted by $Z_3^{(2)}(p-4)$. For $N = 4$, the parafermionic algebra factorizes into a direct product of two $\mathcal{N} = 1$ superconformal algebras. The slightly relevant perturbation on a single $\mathcal{N} = 1$ superconformal algebra has been discussed in [49].

According to the observation of Dotsenko and Estienne [15], the two slightly relevant fields (for odd $N \geq 7$ and for $N = 5$, they also presented the corresponding quantities), can be obtained from the product of WB_n primaries $\Phi_{(\vec{n}|\vec{n}')}^{(p)}$ by decomposing the coset (4.1) into several simpler cosets as follows

$$\begin{aligned} S_{(1^n|3,1^{n-1})}^{(p)} &= \Phi_{(1^n|2,1^{n-1})}^{(p)} \otimes \Phi_{(2,1^{n-1}|3,1^{n-1})}^{(p+1)}, \\ A_{(1^n|1,2,1^{n-2})}^{(p)} &= \frac{1}{\sqrt{2}} \left[\Phi_{(1^n|1^n)}^{(p)} \otimes \Phi_{(1^n|1,2,1^{n-2})}^{(p+1)} + \Phi_{(1^n|1,2,1^{n-2})}^{(p)} \otimes \Phi_{(1,2,1^{n-2}|1,2,1^{n-2})}^{(p+1)} \right]. \end{aligned} \quad (4.3)$$

These two fields appear in the following perturbed action

$$S^{(p)} = S_0^{(p)} + g \int d^2x S_{(1^n|3,1^{n-1})}^{(p)}(x) + h \int d^2x A_{(1^n|1,2,1^{n-2})}^{(p)}(x). \quad (4.4)$$

It is straightforward to compute the conformal dimensions for the fields (Neveu-Schwarz sector) in (4.3) via (3.2)

$$\Delta_{(1^n|2,1^{n-1})}^{(p)} = \frac{(p-N+2)}{2(p+1)} \simeq \frac{1}{2}(1-\lambda),$$

$$\begin{aligned}
\Delta_{(2,1^{n-1}|3,1^{n-1})}^{(p+1)} &= \frac{p(p-N+2)}{2(p+1)(p+2)} \simeq \frac{1}{2}(1-\lambda), \\
\Delta_{(1^n|1^n)}^{(p)} &= 0, \\
\Delta_{(1^n|1,2,1^{n-2})}^{(p+1)} &= \frac{(p-N+4)}{(p+2)} \simeq 1-\lambda, \\
\Delta_{(1^n|1,2,1^{n-2})}^{(p)} &= \frac{(p-N+3)}{(p+1)} \simeq 1-\lambda, \\
\Delta_{(1,2,1^{n-2}|1,2,1^{n-2})}^{(p+1)} &= \frac{(N-2)}{(p+1)(p+2)} \simeq \frac{\lambda^2}{N} \simeq 0.
\end{aligned} \tag{4.5}$$

As expected, the conformal dimensions for $S_{(1^n|3,1^{n-1})}^{(p)}$ and $A_{(1^n|1,2,1^{n-2})}^{(p)}$ in (4.4), in the large (N, k) 't Hooft limit, can be read off from (4.5) and they become $1-\lambda$. The exact expression for the conformal dimension of $A_{(1^n|1,2,1^{n-2})}^{(p)}$ is $1 - \frac{h^\nu}{k+2+h^\nu} = 1 - \frac{N-2}{p+2}$ which can be seen from the result of [26] where there exists only a single relevant field. The first and second representations of (4.1) are trivial representations of $SO(N)$. Moreover, the conformal dimension of $S_{(1^n|3,1^{n-1})}^{(p)}$ is given by $1 - \frac{N}{p+2}$. For large p , they have the same conformal dimension. There exist two kinds of fixed points for nonzero h , which can be seen by analyzing the RG flow from (4.4). Dotsenko and Estienne [15] claim that for the first kind of fixed point, the IR theory is described by $Z_N^{(2)}(p-2)$ parafermion theory while for the second kind of fixed point, the IR theory is given by $Z_N^{(2)}(p-1)$ parafermion theory. The presence of $S_{(1^n|3,1^{n-1})}^{(p)}$ in the perturbed action (4.4) provides the latter critical fixed point. With $A_{(1^n|1,2,1^{n-2})}^{(p)}$ only, the former fixed point occurs.

The deviation of the central charge from the two fixed points can be computed from (4.2) to be

$$\delta c = c_N(p-l) - c_N(p) \simeq -2l\lambda^3, \quad l = 1, 2. \tag{4.6}$$

This can be seen by taking the variation $\delta c = -2N\lambda\delta\lambda$ with $\delta k = -l$ (from $k-l$ to k) in the relation $c_N(\lambda) \simeq N(1-\lambda^2)$. For $l = 1$ in (4.6), the IR theory is given by $Z_N^{(2)}(p-1)$ parafermion theory and for $l = 2$, the IR theory is $Z_N^{(2)}(p-2)$ parafermion theory. One should also see this behavior (4.6) in the bulk. How do the adjoint primary fields (or their WB_n products) flow under the RG flows? Although the particular primary field $\Phi_{(\vec{n}|\vec{n})}^{(p)}$ flows to $\Phi_{(\vec{n}|\vec{n})}^{(p-l)}$ where $l = 1, 2$ under the RG flow [15], it is not known in general how the other primaries flow. It is an open problem to find the gravity duals of the above generalized conformal field theories. For even $N(\geq 6)$, a similar construction is given in [50]. See also [51] for the details. In this case, the constructions (4.3) and (4.5) are based on the $WD_n^{(p)}$ primaries with the conformal dimension formula (2.5). It turns out that the conformal dimensions for $WD_n^{(p)}$ primaries are the same as the ones in (4.5).

Let us discuss the second example where the shift parameter is greater than 1. Although the original motivation of [1] is to search for the nontrivial example of nonsupersymmetric AdS/CFT correspondence, it is an interesting problem to find a supersymmetric version of the proposal of [1]. For example, let us consider the diagonal coset model

$$\frac{\widehat{SU}(N)_3 \oplus \widehat{SU}(N)_k}{\widehat{SU}(N)_{k+3}}. \quad (4.7)$$

The level 3 is crucial for the construction of fermionic currents in order to supersymmetrize the theory. The central charge of (4.7) can be computed from the dual Coxeter number and the dimension of the $SU(N)$ group and is written as

$$\begin{aligned} c_N(p) &= (N^2 - 1) \left[\frac{3}{3+N} + \frac{k}{k+N} - \frac{k+3}{k+3+N} \right] \\ &= \frac{3(N^2 - 1)}{N+3} \left[1 - \frac{N(N+3)}{p(p+3)} \right] \leq \frac{3(N^2 - 1)}{N+3}, \quad p \equiv k+N, k=1, 2, \dots, \end{aligned} \quad (4.8)$$

by considering the right levels. In the large (N, k) 't Hooft limit, this reduces to $c_N(p) \simeq 3N(1 - \lambda^2)$. Again the factor 3 comes from the level of the first group. For $N=3$, the coset constructions and minimal series are found in [52]. The spin $\frac{3}{2}$ fermionic superpartner of $\tilde{T}(z)$, denoted by $\tilde{G}(z)$, can be constructed as in [4] and the spin 3 coset field $\tilde{W}(z)$ can be determined by the requirements [5] that it should be a primary field of dimension 3 with respect to $\tilde{T}(z)$ and the coefficient of the identity in the operator product expansion $\tilde{W}(z)\tilde{W}(w)$ should be $\frac{c}{3}$ with (4.8). Now one can compute the operator product expansion between $\tilde{G}(z)$ and $\tilde{W}(w)$ and it turns out that the spin $\frac{5}{2}$ coset field $\tilde{U}(z)$ is [52, 53]

$$\tilde{U}(z) = d_{abc} \left[\frac{10\lambda^2}{(1-\lambda)(2-\lambda)} \psi_{(1)}^a V_{(2)}^b V_{(2)}^c(z) - \frac{5\lambda}{(1-\lambda)} \psi_{(1)}^a V_{(1)}^b V_{(2)}^c(z) + \psi_{(1)}^a V_{(1)}^b V_{(1)}^c(z) \right], \quad (4.9)$$

where $\psi^a(z)$ is a free fermion field of dimension $\frac{1}{2}$ with $a = 1, 2, \dots, N^2 - 1$ and $V_{(1)}^a(z)$ is a spin 1 current that can be written in terms of free fermions as $V_{(1)}^a(z) = f^{abc}(\psi_{(1)}^b \psi_{(1)}^c)(z)$ up to an overall N -dependent constant with level 3. Similarly, $V_{(2)}(z)$ is a spin 1 current with level k . Here the d_{abc} symbol in (4.9) is the symmetric traceless invariant tensor of rank 3 for $SU(N)$.

Contrary to the description for the spin 3 primary field $\tilde{W}(z)$ [54], for the above spin $\frac{5}{2}$ primary field, there is no vanishing term when we take the large (N, k) 't Hooft limit. This is due to the fact that, by construction, there are no such terms coming from only the second group with subscript (2) and moreover there exists an overall factor $\psi_{(1)}^a(z)$ in (4.9). Since the eigenvalues of the spin 3 mode of the coset algebra corresponding to $\tilde{W}(z)$ in the large

(N, k) 't Hooft limit coincide with the eigenvalues of the zero mode of higher spin 3 in the wedge algebra, one should expect that the above extended currents should preserve the higher spin wedge algebra. The supersymmetric extension of [55] appears in the work of [56, 57]. In the Neveu-Schwarz sector, there is a finite $OSp(1, 2)$ subalgebra generated by the $sl(2)$ generators L_0, L_{\pm} for the Virasoro generator and the mode $G_{\pm\frac{1}{2}}$ of its superpartner. It would be interesting to see how the supersymmetric higher spin algebra [56, 57] is realized in the coset model (4.7) or other unitary coset minimal models.

Other possible supersymmetric versions of [1] can be studied by using the quantum Drinfeld-Sokolov construction of the affine Lie superalgebra $\widehat{SU}(n+1, n)$ that provides the $\mathcal{N} = 2$ super W_n algebras [8].

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